

A two-parametric family of exactly solvable Dirac Hamiltonians

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Abstract

We construct a two-parametric family of exactly solvable Dirac Hamiltonians by the Darboux transformation method. We obtain intertwining relations between different members of the Hamiltonian family. We investigate the spectral properties of the obtained Hamiltonians and the explicit forms of their eigenfunctions.

1 Introduction

The Darboux transformation method [1, 2, 3, 4, 5, 6, 7] is one of the basic methods of construction exactly solvable models of mathematical physics [8, 9, 10, 12, 11, 13, 14, 15, 16, 17].

The idea of the method has been proposed by G. Darboux in 1882 [1]. The method has been developed by M. Crum in 1954 [3].

Modern concept of the Darboux transformation method has been constructed by V. B. Matveev. In 1979 V. B. Matveev generalized and reformulated the Darboux–Crum results for the cases of infinite hierarchies linear and nonlinear differential equations in partial derivatives and some their generalizations (for example, differential–difference and matrix equations), including nonstationary nonlinear Schrödinger, Korteweg–de Vries, Kadomtsev–Petviashvili equations and others [4, 5]. Numerous realizations of the Darboux transformation method [4, 5] have been summarized in monograph [7].

It should be noted, that the results of works [4, 6, 7] contain the cases of stationary and nonstationary two-component Dirac equation. In explicit form the systematic development of the Darboux–Crum transformation method for the one–dimensional two-component Dirac equation is present in [18].

From [7] it is evidently that the Darboux transformations can be applied to the stationary four-component Dirac equation.

In this paper we use the Darboux transformation method for the four–component Dirac equation.

The structure of the present paper is as follows. In Section 2 we generate four one-soliton Dirac Hamiltonians applying the Darboux transformation method for free Dirac Hamiltonians and consider the intertwining relations between the obtained Hamiltonians. In Section 3 we construct two-parameter family of exactly solvable multisoliton solutions. In Section 4 we consider spectrum of the obtained Hamiltonians and the explicit forms of their eigenfunctions.

2 One-soliton Hamiltonians

Let us consider the one-dimensional four-component stationary Dirac equation

$$H_0\psi(x) = E\psi(x), \quad H_0 = -i\alpha_1\partial_x + V_0, \quad \alpha_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad (1)$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Pauli matrix, V_0 is 4×4 Hermitian matrix, $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^t$ is four-component spinor.

Let 4×4 matrix $u(x)$ is the solution of the matrix Dirac equation

$$H_0u(x) = u(x)\Lambda, \quad (2)$$

where Λ is any nonsingular matrix.

Define the Darboux transformation operator

$$L = \frac{\partial}{\partial x} - \frac{du(x)}{dx}u^{-1} \quad (3)$$

and consider the intertwining relation

$$LH_0 = H_1L, \quad (4)$$

then

$$H_1 = -i\alpha_1\partial_x + V_1, \quad V_1 = V_0 + \left[-i\alpha_1, \frac{du(x)}{dx}u^{-1}(x) \right]. \quad (5)$$

The transformed spinor functions

$$\tilde{\psi} = L\psi \quad (6)$$

are solutions of the transformed Dirac equation

$$H_1\tilde{\psi}(x) = E\tilde{\psi}(x). \quad (7)$$

The Darboux transformations of the four-component Dirac equation is similar to the Darboux transformations of the two-component Dirac equation [18].

Consider the free Dirac Hamiltonian H_0 , $V_0 = m\beta$, where m is a mass of particle, $\beta = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}$, $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Suppose that Λ is

$$\Lambda(\lambda_1, \lambda_2) = [\lambda_1(I + \beta) + \lambda_2(I - \beta)]/2, \quad (8)$$

where $\lambda_1 = \pm\varepsilon_0$, $\lambda_2 = \pm\varepsilon_1$, $\varepsilon_n = \sqrt{m^2 - k^2n^2}$, k is real number ($k \ll m$).

Four different sets of parameters λ_1, λ_2 correspond to four different matrices $\Lambda^{(i)}$ ($i = \overline{1, 4}$):

$$\Lambda^{(1)} = \Lambda(\varepsilon_0, \varepsilon_1), \quad \Lambda^{(2)} = \Lambda(\varepsilon_0, -\varepsilon_1), \quad (9)$$

$$\Lambda^{(3)} = \Lambda(-\varepsilon_0, \varepsilon_1), \quad \Lambda^{(4)} = \Lambda(-\varepsilon_0, -\varepsilon_1) \quad (10)$$

and four corresponding transformation matrix functions $u^{(i)}$ are solutions of the matrix Dirac equation

$$H_0u^{(i)} = \Lambda^{(i)}u^{(i)}, \quad i = \overline{1, 4}. \quad (11)$$

All these solutions have the common matrix structure

$$u^{(i)} = a^{(i)}I + b^{(i)}\alpha + c^{(i)}\beta + d^{(i)}\gamma, \quad (12)$$

where

$$\alpha = \alpha_1, \quad \gamma = \alpha\beta, \quad \alpha^2 = \beta^2 = -\gamma^2 = I. \quad (13)$$

It can be shown the inverse matrices have the following general form:

$$\begin{aligned} u^{(i)-1} &= (a^{(i)}I - b^{(i)}\alpha - c^{(i)}\beta - d^{(i)}\gamma)/D, \\ D &= (a^{(i)2} - b^{(i)2} - c^{(i)2} + d^{(i)2}). \end{aligned} \quad (14)$$

where

$$a^{(i)} = \frac{1}{2}(\mu_1^{(i)} + \mu_3^{(i)}), \quad b^{(i)} = \frac{1}{2}(\mu_2^{(i)} + \mu_4^{(i)}), \quad (15)$$

$$c^{(i)} = \frac{1}{2}(\mu_1^{(i)} - \mu_3^{(i)}), \quad d^{(i)} = \frac{1}{2}(\mu_4^{(i)} - \mu_2^{(i)}). \quad (16)$$

Explicit values of magnitudes $\mu_k^{(i)}$

$$\mu_1^{(1)} = \mu_1^{(2)} = 1 = \mu_4^{(3)} = \mu_4^{(4)}, \quad \mu_4^{(1)} = \mu_4^{(2)} = 0 = \mu_1^{(3)} = \mu_1^{(4)}, \quad (17)$$

$$\mu_3^{(1)} = \mu_3^{(2)} = \mu_3^{(3)} = \mu_3^{(4)} = \cosh kx, \quad \mu_2^{(1)} = \frac{ik \sinh kx}{\varepsilon_0 - \varepsilon_1}, \quad (18)$$

$$\mu_2^{(2)} = \frac{ik \sinh kx}{\varepsilon_0 + \varepsilon_1}, \quad \mu_2^{(3)} = -\frac{ik \sinh kx}{\varepsilon_0 + \varepsilon_1}, \quad \mu_2^{(4)} = \frac{ik \sinh kx}{-\varepsilon_0 + \varepsilon_1}. \quad (19)$$

Constructing the Darboux transformation operators with the help of the obtained above matrices

$$L^{(i)} = \frac{d}{dx} - \frac{du^{(i)}}{dx}(u^{(i)})^{-1}, \quad (20)$$

and the intertwining relations

$$L^{(i)}H_0^{(i)} = H_1^{(i)}L^{(i)} \quad (21)$$

we generate four new Hamiltonians

$$H_1^{(i)} = H_0^{(i)} + \left[-i\alpha, \frac{du^{(i)}}{dx}(u^{(i)})^{-1} \right], \quad i = \overline{1, 4}, \quad (22)$$

$$H_1^{(1)} = -i\alpha_1\partial_x - \varepsilon_1\beta + ik \tanh(kx)\gamma, \quad (23)$$

$$H_1^{(2)} = -i\alpha_1\partial_x + \varepsilon_1\beta + ik \tanh(kx)\gamma, \quad (24)$$

$$H_1^{(3)} = -i\alpha_1\partial_x - \varepsilon_1\beta - ik \tanh(kx)\gamma, \quad (25)$$

$$H_1^{(4)} = -i\alpha_1\partial_x + \varepsilon_1\beta - ik \tanh(kx)\gamma. \quad (26)$$

The Hamiltonians $H_1^{(1)}$, $H_1^{(3)}$, and $H_1^{(2)}$, $H_1^{(4)}$, are related by unitary transformation

$$H_1^{(3)} = UH_1^{(1)}U^{-1}, \quad H_1^{(4)} = UH_1^{(2)}U^{-1}, \quad U = \alpha \quad (27)$$

and consequently they are isospectral.

Spectra of the Hamiltonians $H_1^{(1)}$, $H_1^{(3)}$, $H_1^{(2)}$, $H_1^{(4)}$, in difference of $H^{(0)}$ contain both two continuous branches $m \leq E < \infty$, $-\infty < E \leq -m$ and discrete part containing one level. Energies of bind states of the Hamiltonians $H_1^{(1)}$ and $H_1^{(3)}$ are equal ε_1 . Energies of bind states of the Hamiltonians $H_1^{(2)}$, $H_1^{(4)}$ are equal $-\varepsilon_1$.

Besides of relations (27), between Hamiltonian pairs $H_1^{(i)}$ there are intertwining relations

$$L_1^{(1,4)} H_1^{(1)} = H^{(4)} L_1^{(1,4)}, \quad L_1^{(2,3)} H_1^{(2)} = H^{(3)} L_1^{(2,3)}, \quad (28)$$

$$L_1^{(4,1)} H_1^{(4)} = H_1^{(1)} L_1^{(4,1)}, \quad L_1^{(3,2)} H_1^{(3)} = H_1^{(3)} L_1^{(3,2)}, \quad (29)$$

$$L_1^{(1,2)} H_1^{(1)} = H_1^{(2)} L_1^{(1,2)}, \quad L_1^{(2,1)} H_1^{(2)} = H_1^{(1)} L_1^{(2,1)}, \quad (30)$$

$$L_1^{(3,4)} H_1^{(3)} = H^{(4)} L_1^{(3,4)}, \quad L_1^{(4,3)} H_1^{(4)} = H^{(3)} L_1^{(4,3)}. \quad (31)$$

Here

$$L_1^{(1,4)} = L_1^{(2,3)} = \bar{L}_1, \quad L_1^{(4,1)} = L_1^{(3,2)} = \tilde{L}_1, \quad (32)$$

$$L_1^{(1,2)} = \bar{L}_1 U, \quad L_1^{(2,1)} = \tilde{L}_1 U, \quad (33)$$

$$L_1^{(3,4)} = \bar{L}_1 U, \quad L_1^{(4,3)} = \tilde{L}_1 U, \quad (34)$$

$$\bar{L}_1 = \frac{d}{dx} - \frac{d\bar{u}}{dx}(\bar{u})^{-1}, \quad \tilde{L}_1 = \frac{d}{dx} - \frac{d\tilde{u}}{dx}(\tilde{u})^{-1}, \quad (35)$$

where

$$\bar{u} = \cosh(kx)(I + \beta) + \frac{1}{\cosh(kx)}(I - \beta), \quad (36)$$

$$\tilde{u} = \cosh(kx)(\alpha + \gamma) + \frac{1}{\cosh(kx)}(\alpha - \gamma). \quad (37)$$

3 Multi-soliton shape-invariant Hamiltonians

Consider the following Hamiltonians $H_n^{(i)}$ ($i = \overline{1, 4}$)

$$H_n^{(1)} = -i\alpha_1 \frac{d}{dx} - \varepsilon_n \beta + ink \tanh(kx) \gamma, \quad (38)$$

$$H_n^{(2)} = -i\alpha_1 \frac{d}{dx} + \varepsilon_n \beta + ink \tanh(kx) \gamma, \quad (39)$$

$$H_n^{(3)} = -i\alpha_1 \frac{d}{dx} + \varepsilon_n \beta - ink \tanh(kx) \gamma, \quad (40)$$

$$H_n^{(4)} = -i\alpha_1 \frac{d}{dx} - \varepsilon_n \beta - ink \tanh(kx) \gamma, \quad (41)$$

coinciding with the obtained Hamiltonians $H_1^{(i)}$ at $n = 1$.

It is evidently that

$$H_n^{(3)} = U H_n^{(1)} U^{-1}, \quad H_n^{(4)} = U H_n^{(2)} U^{-1}, \quad U = \alpha_1. \quad (42)$$

It is easily to calculate that the intertwining relations $L_n^{(i,k)} H_n^{(i)} = H_{n+1}^{(k)} L_n^{(i,k)}$ ($i, k = 1, 2$ or $i, k = 3, 4$) have place.

At that

$$\begin{aligned}\bar{L}_n H_n^{(1)} &= H_n^{(4)} \bar{L}_n, & \tilde{L}_n H_n^{(4)} &= H_n^{(1)} \tilde{L}_n, \\ \bar{L}_n H_n^{(2)} &= H_n^{(3)} \bar{L}_n, & \tilde{L}_n H_n^{(3)} &= H_n^{(2)} \tilde{L}_n,\end{aligned}\tag{43}$$

where

$$\bar{L}_n = \frac{d}{dx} - \frac{d\bar{u}_n}{dx}(\bar{u}_n)^{-1}, \quad \tilde{L}_n = \frac{d}{dx} - \frac{d\tilde{u}_n}{dx}(\tilde{u}_n)^{-1},\tag{44}$$

$$\bar{u}_n = \cosh^n(kx)(I + \beta) + \coth^{-n}(kx)(I - \beta),\tag{45}$$

$$\tilde{u}_n = \cosh^n(kx)(\alpha + \gamma) + \coth^{-n}(kx)(\alpha - \gamma),\tag{46}$$

$$L_n^{(i,k)} = \frac{d}{dx} - \frac{d}{dx}(u_n^{(i,k)})(u_n^{(i,k)})^{-1},\tag{47}$$

$$u_n^{(i,k)} = a_n^{(i,k)}I + b_n^{(i,k)}\alpha + c_n^{(i,k)}\beta + d_n^{(i,k)}\gamma,\tag{48}$$

$$a_n^{(i,k)} = \frac{1}{2}(\mu_{n,1}^{(i,k)} + \mu_{n,3}^{(i,k)}), \quad b_n^{(i,k)} = \frac{1}{2}(\mu_{n,2}^{(i,k)} + \mu_{n,4}^{(i,k)}),\tag{49}$$

$$c_n^{(i,k)} = \frac{1}{2}(\mu_{n,1}^{(i,k)} - \mu_{n,3}^{(i,k)}), \quad d_n^{(i,k)} = \frac{1}{2}(\mu_{n,4}^{(i,k)} - \mu_{n,2}^{(i,k)}).\tag{50}$$

The obvious expressions for quantities $\mu_{n,j}^{(i,k)}$ ($j = \overline{1,4}$) are presented in Appendix.

The transformation matrixes $u_n^{(i,k)}$ are solutions of the matrix Dirac equations

$$H_n^{(i)} u_n^{(i,k)} = u_n^{(i,k)} \Lambda_n^{(i,k)}, \quad \Lambda_n^{(i,k)} = \frac{1}{2} \left[(I + \beta) \lambda_{n,1}^{(i,k)} + (I - \beta) \lambda_{n,2}^{(i,k)} \right].\tag{51}$$

The values of quantities $\lambda_{n,j}^{(i,k)}$ ($j = 1, 2$) are in Appendix.

Intertwining relations (43) are presented in the diagram form

$$\begin{array}{ccccc} & L_0^{(2)} & & & \\ & \nearrow & & & \\ H_0 & H_1^{(2)} & \xrightarrow{L_1^{(2)}} & H_2^{(2)} & \xrightarrow{L_2^{(2)}} \dots \xrightarrow{L_{n-1}^{(2)}} H_n^{(2)} \xrightarrow{L_n^{(2)}} H_{n+1}^{(2)} \dots \\ & L_1^{(1,2)} & \nearrow & \searrow & L_2^{(2,1)} \quad L_n^{(1,2)} \nearrow \\ & L_0^{(1)} & & & \\ & \searrow & & & \\ & H_1^{(1)} & \xrightarrow{L_1^{(1)}} & H_2^{(1)} & \xrightarrow{L_2^{(1)}} \dots \xrightarrow{L_{n-1}^{(1)}} H_n^{(1)} \xrightarrow{L_n^{(1)}} H_{n+1}^{(1)} \dots \\ & \tilde{L}_2 \uparrow \downarrow \tilde{L}_2 & & & \tilde{L}_n \uparrow \downarrow \tilde{L}_n \\ & L_0^{(4)} & & & \\ & \nearrow & & & \\ H_0 & H_1^{(4)} & \xrightarrow{L_1^{(4)}} & H_2^{(4)} & \xrightarrow{L_2^{(4)}} \dots \xrightarrow{L_{n-1}^{(4)}} H_n^{(4)} \xrightarrow{L_n^{(4)}} H_{n+1}^{(4)} \dots \\ & L_1^{(3)} & \nearrow & \searrow & L_2^{(3)} \quad L_n^- \nearrow \\ & L_0^{(3,4)} & & & \\ & \searrow & & & \\ & H_1^{(4,3)} & \xrightarrow{L_1^{(3,4)}} & H_2^{(3)} & \xrightarrow{L_2^{(3)}} \dots \xrightarrow{L_{n-1}^{(3)}} H_n^{(3)} \xrightarrow{L_n^{(3)}} H_{n+1}^{(3)} \dots \end{array}$$

Hence with the help of the Darboux transformation technique (the Darboux–Crum chains [3]) we construct two-parameter family of the Dirac Hamiltonians $H_n^{(i)}$, $i = \overline{1,4}$, which are $(2n-1)$ -solitons, i. e. they have $2n-1$ bound states.

4 Spectral structure and eigenfunctions of Hamiltonians $H_n^{(i)}$

To investigate the solutions of the equations

$$H_n^{(i)}\psi^{(i)} = E_n\psi^{(i)}, \quad i = 1, 2, \quad (52)$$

we have

$$\psi^{(3)} = \alpha\psi^{(1)}, \quad \psi^{(4)} = \alpha\psi^{(2)}. \quad (53)$$

From (52) and (38), (39) it is easily to obtain that the components of spinors

$$\psi^{(i)} = (\psi_1^{(i)}, \psi_2^{(i)}, \psi_3^{(i)}, \psi_4^{(i)})^T$$

at $i = 1, 2$ are the solutions of second-order equations

$$(\psi_j^{(i)})'' + \frac{n(n-1)k^2}{\cosh^2 kx} \psi_j^{(i)} + (E^2 - m^2)\psi_j^{(i)} = 0, \quad j = 1, 2, \quad (54)$$

$$(\psi_j^{(i)})'' + \frac{n(n+1)k^2}{\cosh^2 kx} \psi_j^{(i)} + (E^2 - m^2)\psi_j^{(i)} = 0, \quad j = 3, 4, \quad (55)$$

which with the help of substitution $t = \tanh kx$ are reduced to the equations for associated Legendre polynomials.

The energy spectrum at $|E| \geq m$ is continuous and the energy spectrum at $|E| < m$ is discrete. The energy levels are

$$E_s = \text{sign}(s)\varepsilon_s = \text{sign}(s)\sqrt{m^2 - k^2 s^2}, \quad (56)$$

where

$$s = \begin{cases} -n+1, -n+2, \dots, -1, 1, \dots, n-1, n, & i = 1, \\ -n, -n+1, \dots, -1, 1, \dots, n-2, n-1, & i = 2. \end{cases} \quad (57)$$

The solutions of equations

$$H_n^{(1)}\psi_s = E_s\psi_s^{(1)} \quad (58)$$

are

$$\psi_s^{(1)} = (\psi_{s1}^{(1)}, \psi_{s2}^{(1)}, \psi_{s3}^{(1)}, \psi_{s4}^{(1)})^T, \quad (59)$$

$$\psi_{s1}^{(1)} = A_s^{(1)}P_{n-1}^s(t), \quad \psi_{s2}^{(1)} = B_s^{(1)}P_{n-1}^s(t), \quad (60)$$

$$\psi_{s3}^{(1)} = C_s^{(1)}P_n^s(t), \quad \psi_{s4}^{(1)} = D_s^{(1)}P_n^s(t), \quad t = \tanh kx, \quad (61)$$

$$A_s^{(1)} = \frac{ik(n+s)}{E_s + \varepsilon_n}D_s^{(1)}, \quad B_s^{(1)} = \frac{ik(n+s)}{E_s + \varepsilon_n}C_s^{(1)}. \quad (62)$$

The solutions of equations

$$H_n^{(2)}\psi_s = E_s\psi_s^{(2)} \quad (63)$$

are

$$\psi_s^{(2)} = (\psi_{s1}^{(2)}, \psi_{s2}^{(2)}, \psi_{s3}^{(2)}, \psi_{s4}^{(2)})^T, \quad (64)$$

$$\psi_{s1}^{(2)} = A_s^{(2)}P_{n-1}^s(t), \quad \psi_{s2}^{(2)} = B_s^{(2)}P_{n-1}^s(t), \quad (65)$$

$$\psi_{s3}^{(2)} = C_s^{(2)}P_n^s(t), \quad \psi_{s4}^{(2)} = D_s^{(2)}P_n^s(t), \quad t = \tanh kx, \quad (66)$$

$$A_s^{(2)} = \frac{ik(n-s)}{E_s - \varepsilon_n}D_s^{(2)}, \quad B_s^{(2)} = \frac{ik(n-s)}{E_s - \varepsilon_n}C_s^{(2)}. \quad (67)$$

Here $P_{n-1}^s(t)$ and $P_n^s(t)$ are associated Legendre polynomials [22].

Continuous spectrum is doubly degenerate, i. e. for each value E ($|E| \geq m$) equations (52) have two linear-independent solutions, that can be obtained from the wave functions of discrete spectrum by substitution

$$E_s \rightarrow E, \quad s \rightarrow \pm i\mu, \quad P_n^s \rightarrow P_n^{\pm i\mu}(t), \quad (68)$$

$$P_n^{\pm i\mu}(t) = \frac{1}{\Gamma(1 \mp i\mu)} \exp(\pm ipx) F(-n, n+1; 1 \mp i\mu; (1-t)/2). \quad (69)$$

From the last expression it is evident that the obtained potentials are transparent.

5 Conclusion

In the paper we constructed the special Darboux transformation chains. These chains generate the set of the shape invariant exactly solvable four-component Dirac Hamiltonians. The Hamiltonians may be applied for investigation of spin 1/2 relativistic particle moving in one-dimensional periodic structures [23, 24, 25]. The results corresponding investigations will be published.

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Appendix

In this Appendix we demonstrate the explicit forms

(i) for quantities $\lambda_{n,j}^{(i,k)}$:

$$\lambda_{n,1}^{(1,1)} = \lambda_{n,1}^{(1,2)} = \lambda_{n,1}^{(3,3)} = \lambda_{n,1}^{(3,4)} = -\varepsilon_n, \quad (70)$$

$$\lambda_{n,1}^{(2,2)} = \lambda_{n,1}^{(2,1)} = \lambda_{n,1}^{(4,4)} = \lambda_{n,1}^{(4,3)} = \varepsilon_n, \quad (71)$$

$$\lambda_{n,2}^{(1,1)} = \lambda_{n,2}^{(2,1)} = \lambda_{n,2}^{(3,3)} = \lambda_{n,2}^{(4,3)} = \varepsilon_{n+1}, \quad (72)$$

$$\lambda_{n,2}^{(2,2)} = \lambda_{n,2}^{(1,2)} = \lambda_{n,2}^{(4,4)} = \lambda_{n,2}^{(3,4)} = -\varepsilon_{n+1}, \quad (73)$$

(ii) and also for quantities $\mu_{n,j}^{(i,k)}$ ($i, k = 1, 2$) :

$$\mu_{n,1}^{(i,k)} = \cosh^n kx, \quad \mu_{n,2}^{(i,k)} = \frac{(2n+1)ik \sinh kx \cosh^n kx}{\lambda_{n,1}^{(i,k)} - \lambda_{n,2}^{(i,k)}}, \quad (74)$$

$$\mu_{n,3}^{(i,k)} = \cosh^{n+1} kx, \quad \mu_{n,4}^{(i,k)} = 0, \quad (75)$$

(iii) and $\mu_{n,j}^{(i,k)}$ ($i, k = 3, 4$) :

$$\mu_{n,1}^{(i,k)} = 0, \quad \mu_{n,2}^{(i,k)} = \cosh^{n+1} kx, \quad (76)$$

$$\mu_{n,3}^{(i,k)} = \frac{(2n+1)ik \sinh kx \cosh^n kx}{\lambda_{n,1}^{(i,k)} - \lambda_{n,2}^{(i,k)}}, \quad \mu_{n,4}^{(i,k)} = \cosh^n kx. \quad (77)$$

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